

Block Thresholding on the Sphere

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Abstract

The aim of this paper is to study the nonparametric regression estimators on the sphere built by the needlet block thresholding. The block thresholding procedure proposed here follows the method introduced by Hall, Kerkycharian and Picard in [24], [25], modified to exploit the properties of the spherical standard needlets. Therefore, we will investigate on their convergence rates, attaining their adaptive properties over the Besov balls. This work is strongly motivated by issues arising in Cosmology and Astrophysics, concerning in particular the analysis of cosmic rays.

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1 Introduction

Over the last years, wavelet techniques have been used to achieve remarkable results in the field of statistics, in particular in the framework of minimax estimation in nonparametric settings. The pioneering work in this area was provided by Donoho et al. in [11], where authors proved that nonlinear wavelet estimators based on thresholding techniques attain nearly optimal minimax rates, up to logarithmic terms, for a large class of unknown density and regression functions. Since then, this research area has been deeply investigated and extended - we suggest for instance [23] as a textbook reference: specifically, we are focussing on the wavelet block thresholding procedure, among the other techniques. Loosely speaking, this method keeps or annihilates blocks of wavelet coefficients on each given level (for more details, see [23]), hence representing an intermediate way between local and global thresholding, which fix a threshold respectively for each coefficient and for all the set of them. This procedure, initially suggested in [15] for orthogonal series based estimators and later applied by [24] for both wavelet and kernel density estimation on \mathbb{R} (see also [25]), was used in [5] jointly

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to Oracle inequalities; overlapping block thresholding estimators were studied in [6]. The block thresholding was also applied to study adaptivity in density estimation in [9], a data-driven block thresholding procedure for wavelet regression is instead investigated in [7], while wavelet-based block thresholding rules on maxisets are proposed by [1].

Even if a huge number of results concerns estimation with the thresholding paradigm in standard Euclidean frameworks, such as \mathbb{R} or \mathbb{R}^n , more recent applications are being established in more general settings, such as spherical data or more general manifolds. In particular, we aim to a highly successful construction of a second-generation wavelet system on the sphere, the so-called needlets. The needlets were introduced by Narcowich, Petrushev and Ward in [36], [37]; their stochastic properties, when exploited on spherical random fields, were studied in [2], [3], [32] and [33]. This approach has been extended to more general manifolds by [20], [21], [22], while their generalization to spin fiber bundles on the sphere were described in [18], [19]. Most of these researches can be motivated in view of their applications to Cosmology and Astrophysics: for instance, a huge amount of spherical data, concerning the Cosmic Microwave Background radiation, are being provided by satellite missions WMAP and Planck, see [38], [35], [39], [16], [40], [41], [10], [42], [13] and [14] for more details. The applications mentioned here, however, do not concern thresholding estimation, but rather they can be related to the study of random fields on the sphere, such as angular power spectrum estimation, higher-order spectra, testing for Gaussianity and isotropy, and several others (see also [8]). As another example, we mention experiments concerning incoming directions of Ultra High Energy Cosmic Rays, such as the AUGER Observatory (<http://www.auger.org>). The Ultra-High Energy Cosmic Rays are particles with energy above 10^{18} eV reaching the Earth. Even if they were discovered almost a century ago, their origin, their mechanisms of acceleration and propagation are still unknown. As described in [4], see also [17], an efficient nonparametric estimation of the density function of these data would explain the origin of the High Energy Cosmic Rays, i.e. if it is uniform, they are generated by cosmological effects, such as the decay of the massive particles generated during the Big Bang, or, on the other hand, if it is highly non-uniform and, moreover, strongly correlated with the local distribution of nearby Galaxies, it means that the Cosmic Rays are generated by astrophysical phenomena, as for instance acceleration into Active Galactic Nuclei. Massive amount of data in this area are expected to be available in the next few years. Also in view of this application, the needlet approach was recently applied within the thresholding paradigm to the estimation of the directional data: the seminal contribution in this field is due to [4], see also [28], [27], while applications to astrophysical data is still under way, see for instance [16], [17] and [26]. Minimax estimators for spherical data, outside the needlets approach, were also studied by Kim and coauthors (see [30], [29], [31]). Furthermore, adaptive nonparametric regression estimators of spin-functions, based on spin pure and mixed needlets defined in [18], [19], were investigated in [12]. In this case, the needlet nonparametric regression estimators were built on spin fiber bundles on the sphere, i. e. the function to be estimated does not take as

its values scalars but algebraic curves living on the tangent plane for each point of the sphere.

This work, hence, extends the results established in [4] and [12] towards the needlet block thresholding procedure following two main directions. First of all, we will suggest a construction of blocks of needlet coefficients, exploiting the Voronoi cells based on the geodesic distance on the sphere. Then, we will define the needlet block thresholding estimator, whose we will achieve an optimal convergence rate. In view of this aim, we will use both the needlet properties established in [36], [37] (see also [34]) and a set of well consolidated standard techniques, introduced by [11] (see also [23]), remarking that this approach has been also applied in the needlet framework to local thresholding by [4] and [12].

Section 2 will recall some preliminary notions, as needlets, their main properties and the Besov spaces. Section 3 will describe the block thresholding procedure we suggest for needlet regression estimation, while Section 4 will present the main minimax results. Section 5 will collect some useful auxiliary results, while Section 6 will exploit the proof of the main result of this work, named as Theorem 1.

2 Background results

In this Section, we will review briefly a few of well-known features about the Voronoi cells on the sphere, the spherical needlet construction and the Besov spaces.

For what concerns Voronoi cells, we are following strictly [3]: further details can be found for instance in the textbook [34], see also [2] and [37]. From now on, given two positive sequences $\{a_j\}$ and $\{b_j\}$, we write that $a_j \approx b_j$ if there exists a constant $c > 0$ so that $c^{-1}a_j \leq b_j \leq ca_j$ for all j . Furthermore, $B_{x_0}(\alpha) = \{x \in \mathbb{S}^2 : d(x, x_0) < \alpha\}$ and $\overline{B}_{x_0}(\alpha) = \{x \in \mathbb{S}^2 : d(x, x_0) \leq \alpha\}$ denote respectively standard open and closed balls on \mathbb{S}^2 around $x_0 \in \mathbb{S}^2$, while $|A|$ is the spherical measure of a general subset $A \subset \mathbb{S}^2$. Given $\varepsilon > 0$, the set $\Xi_\varepsilon = \{x_1, \dots, x_N\}$ of points on \mathbb{S}^2 , such that for $i \neq j$ we have $d(x_i, x_j) > \varepsilon$, is called a *maximal ε -net* if it satisfies $d(x, \Xi_\varepsilon) < \varepsilon$ for $x \in \mathbb{S}^2$, $\cup_{x_i \in \Xi_\varepsilon} B_{x_i}(\varepsilon) = \mathbb{S}^2$ and $B_{x_i}(\varepsilon/2) \cap B_{x_j}(\varepsilon/2) = \emptyset$, for $i \neq j$. For all $x_i \in \Xi_\varepsilon$, a family of Voronoi cells is defined as

$$\mathcal{V}(x_i) = \{x \in \mathbb{S}^2 : \text{for } j \neq i, d(x, x_i) < d(x, x_j)\}. \quad (1)$$

In [3] it is proved that:

$$B_{x_i}\left(\frac{\varepsilon}{2}\right) \subset \mathcal{V}(x_i) \subset B_{x_i}(\varepsilon) .$$

We resume now some features on the scalar needlet construction, system, suggesting for a more detailed discussion [36], [37], see also [4] and [34]. A needlet system describes a well-localized tight frame on the sphere: it is a well-known fact (cfr. [36]) that any function belonging to $L^2(\mathbb{S}^2)$ can be represented as a linear combination of the components of that frame, preserving furthermore

some fundamental properties of needlets. Indeed, let us recall that the space $L_2(\mathbb{S}^2)$ of square-integrable functions on the sphere can be decomposed as the direct sum of the spaces H_l of harmonic polynomials of degree l , spanned by spherical harmonics $\{Y_{lm}\}_{m=-l}^l$, whose definition and properties can be found in [43] and [4]. If we consider

$$\Pi_l = \bigoplus_{l'=0}^l H_{l'},$$

the space of the restrictions to \mathbb{S}^2 of the polynomials of degree less (and equal) to l , the following quadrature formula holds (see for instance [4]): given $l \in \mathbb{N}$, there exists a finite subset χ_l such that a positive real number λ_ξ (the cubature weight) corresponds to each $\xi \in \chi_l$ (the cubature point) and for all $f \in \Pi_l$,

$$\int_{\mathbb{S}^2} f(x) dx = \sum_{\xi \in \chi_l} \lambda_\xi f(\xi).$$

Given $B > 1$ and a resolution level j , we call $\chi_{[B^{2(j+1)}]} = \mathcal{Z}_j$, $\text{card}(\mathcal{Z}_j) = N_j$; since now any element of the set of cubature points and weights, $\{\xi_{jk}, \lambda_{jk}\}$, will be indexed by j , the resolution level, and k , the cardinality over j , belonging to \mathcal{Z}_j . Furthermore, we choose $\{\mathcal{Z}_j\}_{j \geq 1}$ to be nested so that

$$N_j \approx B^{2j}, \lambda_{jk} \approx B^{-2j}. \quad (2)$$

We consider a symmetric, real-valued, not negative function $b(\cdot)$ (see again [4]) such that

1. it has compact support on $[B^{-1}, B]$;
2. $b \in C^\infty(\mathbb{R})$;
3. the following *unitary property* holds for $|\xi| \geq 1$:

$$\sum_{j \geq 0} b^2\left(\frac{\xi}{B^j}\right) = 1.$$

For each $\xi_{jk} \in \mathcal{Z}_j$, given $b(\cdot)$ and B , scalar needlets are defined as:

$$\psi_{jk}(x) = \sqrt{\lambda_{jk}} \sum_{B^{j-1} < l < B^{j+1}} b\left(\frac{l}{B^j}\right) L_l(\langle x, \xi_{jk} \rangle),$$

where $L_l(\langle x, y \rangle) = \sum_{m=-l}^l Y_{lm}(x) \overline{Y_{lm}(y)}$, describing loosely speaking the weighted convolution of the projection operator $L_l(\langle x, y \rangle)$.

The properties of the function $b(\cdot)$ reflect three basic features of the needlets. Indeed, from the infinite differentiability of $b(\cdot)$, we have a quasi-exponential

localization property (see for instance [37]), which states that for $k \in \mathbb{N}$, there exists c_k such that for $x \in \mathbb{S}^2$

$$|\psi_{jk}(x)| \leq \frac{c_k B^j}{(1 + B^j d(\xi_{jk}, x))^k}, \quad (3)$$

where $d(\xi_{jk}, x)$ is the geodesic distance on the sphere. In view of this property, it is possible to fix a bound (upper and lower), for the norms of needlets on $L^p(\mathbb{S}^2)$, for $1 \leq p \leq +\infty$. Given p , there exist two positive constants c_p and C_p such that

$$c_p B^{j(1-\frac{2}{p})} \leq \|\psi_{jk}\|_{L^p(\mathbb{S}^2)} \leq C_p B^{j(1-\frac{2}{p})}. \quad (4)$$

Because the function $b(\cdot)$ has compact support in $[B^{-1}, B]$, it follows that $b(\frac{l}{B^j})$ has compact support in $[B^{j-1}, B^{j+1}]$, hence needlets have compact support in the harmonic domain.

Finally, the unitary property leads to the following reconstruction formula (see again [36]): for $f \in L^2(\mathbb{S}^2)$, in the L^2 sense,

$$f(x) = \sum_{j,k} \beta_{jk} \psi_{jk}(x), \quad (5)$$

$$\beta_{jk} := \langle f, \psi_{jk} \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \overline{\psi_{jk}}(x) f(x) dx, \quad (6)$$

where β_{jk} are the so-called needlet coefficients.

Before concluding this Section, we recall the definition and some main properties of the Besov spaces, referring again to [4], [12] and [23] for further theoretical details and discussions. Let $f \in L_\pi(\mathbb{S}^2)$; we define

$$G_k(f, \pi) = \inf_{H \in \mathcal{H}_k} \|f - H\|_{L_\pi(\mathbb{S}^2)},$$

which is the approximation error when replacing f by an element in \mathcal{H}_k . The Besov space $\mathcal{B}_{\pi q}^r$ is therefore defined as the space of functions such that $f \in L_\pi(\mathbb{S}^2)$ and

$$\left(\sum_{k=0}^{\infty} \frac{1}{k} (k^r G_k(f, \pi))^q \right) < \infty.$$

The last condition is equivalent to

$$\left(\sum_{j=0}^{\infty} (B^{jr} G_{B^j}(f, \pi))^q \right) < \infty.$$

Moreover, $F \in \mathcal{B}_{\pi q}^r$ if and only if, for every $j = 1, 2, \dots$

$$\left(\sum_k \left(|\beta_{jk}| \|\psi_{jk}\|_{L_\pi(\mathbb{S}^2)} \right)^\pi \right)^{\frac{1}{\pi}} = \varepsilon_j B^{-jr}$$

where $\varepsilon_j \in \ell_q$ and $B > 1$. The Besov norm is defined as follows:

$$\|f\|_{\mathcal{B}_{\pi q}^r} = \begin{cases} \|f\|_{L_\pi(\mathbb{S}^2)} + \left[\sum_j B^{jq(r+\frac{1}{2}-\frac{1}{\pi})} \left\{ \sum_k |\beta_{jk}|^\pi \right\}^{\frac{q}{\pi}} \right]^{\frac{1}{q}} & q < \infty \\ \|f\|_{L_\pi(\mathbb{S}^2)} + \sup_j B^{jq(r+\frac{1}{2}-\frac{1}{\pi})} \left\| (\beta_{jk})_k \right\|_{\ell_\pi} & q = \infty \end{cases}.$$

As shown for instance in [4], if $\max(0, 1/\pi - 1/q) < r$ and $\pi, q > 1$, then we have

$$f \in \mathcal{B}_{\pi q}^r \Leftrightarrow \|f\|_{\mathcal{B}_{\pi q}^r} < \infty.$$

The Besov spaces present, among their properties, some embeddings which will be pivotal in our proofs below. As proven in [4] and [12], we have that, for $\pi_1 \leq \pi_2$, $q_1 \leq q_2$

$$\mathcal{B}_{\pi q_1}^r \subset \mathcal{B}_{\pi q_2}^r, \mathcal{B}_{\pi_2 q}^r \subset \mathcal{B}_{\pi_1 q}^r, \mathcal{B}_{\pi_1 q}^r \subset \mathcal{B}_{\pi_2 q}^{r-\frac{1}{\pi_1}+\frac{1}{\pi_2}}. \quad (7)$$

3 Needlet Block Thresholding on the Sphere

In this Section we will present needlet estimators for nonparametric regression problems and, then, we will suggest a procedure to fix blocks for any given resolution level j and, consequently, we will define the so-called needlet block threshold estimator. The first step is close to the one described in [4], [12] for local thresholding, the other one being an adaptation to the sphere of the procedure developed for \mathbb{R} in [24], [25], see also [23]

In order to introduce the nonparametric regression estimator, let us initially define the so-called uncentered isonormal Gaussian process with mean f (see *****). More precisely, we assume to have a family of Gaussian variables such that for all $h_1, h_2 \in \mathfrak{H}$, $\{X(h_1), X(h_2)\}$ are jointly Gaussian with mean

$$\mathbb{E}X(h) = \langle h, f \rangle = \int_{S^2} f(x)h(x)dx$$

and covariance

$$\mathbb{E}(X(h_1) - \mathbb{E}X(h_1))(X(h_2) - \mathbb{E}X(h_2)) = \langle h_1, h_2 \rangle_{L^2(S^2)}.$$

We shall in fact be concerned with sequences $\{X_n\}$ of such processes, where we assume that

$$\mathbb{E}X_n(h) = \langle h, f \rangle = \int_{S^2} f(x)h(x)dx$$

and covariance

$$\mathbb{E}(X_n(h_1) - \mathbb{E}X_n(h_1))(X_n(h_2) - \mathbb{E}X_n(h_2)) = \frac{1}{n} \langle h_1, h_2 \rangle_{L^2(S^2)}.$$

Consider now the usual needlet system $\{\psi_{jk}\}_{j,k}$ and let $f \in L^p(S^2)$; we have the following:

$$\begin{aligned}\beta_{jk} &= \mathbb{E}X_n(\psi_{jk}) = \langle \psi_{jk}, f \rangle = \int_{S^2} f(x) \psi_{jk}(x) dx , \\ \widehat{\beta}_{jk} &= X_n(\psi_{jk}) = \beta_{jk} + \varepsilon_{jk;n} ,\end{aligned}\tag{8}$$

where

$$\begin{aligned}\mathbb{E}\varepsilon_{jk;n} &= \mathbb{E}(X_n(\psi_{jk}) - \mathbb{E}X_n(\psi_{jk})) = 0 , \\ E\varepsilon_{jk;n}^2 &= \frac{1}{n} \langle \psi_{jk}, \psi_{jk} \rangle_{L^2(S^2)} = \frac{1}{n} \|\psi_{jk}\|_{L^2(S^2)}^2 , \\ E\varepsilon_{jk_1;n} \varepsilon_{jk_2;n} &= \frac{1}{n} \langle \psi_{jk_1}, \psi_{jk_2} \rangle_{L^2(S^2)} \\ &= \frac{1}{n} \frac{\sum_l b^2(\frac{l}{2j})^{\frac{2l+1}{4\pi}} P_l(\langle \xi_{jk_1}, \xi_{jk_2} \rangle)}{\sum_l b^2(\frac{l}{2j})^{\frac{2l+1}{4\pi}}} .\end{aligned}\tag{9}$$

In a formal sense, one could consider the Gaussian white noise measure on the sphere such that for all $A, B \subset S^2$, we have

$$\mathbb{E}W(A)W(B) = \int_{A \cap B} dx ,$$

so that

$$\varepsilon_{jk;n} = \frac{1}{n} \int_{S^2} \psi_{jk}(x) W(dx) .$$

As described above (see also [4], [12]), f can be described in terms of needlet coefficients, up to a constant, as

$$f = \sum_{j \geq 0} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} .$$

Let us now define the blocks on which we will apply the thresholding procedure: as anticipated in the Introduction, differently from [24], the structure itself of the needlet framework puts in evidence a way to build them. Let us fix $j > 0$: recall that for each resolution level j , we have $N_j \approx B^{2j}$ cubature points. Given the size of the blocks, i.e. the number of cubature points belonging to each of them - let us say ℓ_j - we will build using (1) a set of Voronoi cells, containing ℓ_j cubature points. For each cell, we choose a cubature point ξ_{js} to index it: we define $S_j(\ell_j)$ as the number of Voronoi cells obtained to split cubature points into groups of cardinality ℓ_j . Let us define the set

$$R_{j;s} = \{k : \xi_{jk} \in \mathcal{V}(\xi_{js})\} , \quad s = 1, \dots, S_j .$$

From (1), it is immediate to see that each cubature point ξ_{jk} belongs to a unique Voronoi cell. We choose ℓ_j such that

$$\ell_j = \lceil N_j^\eta \rceil \approx N_j^\eta .$$

where $[\cdot]$ denotes the integer part and $0 < \eta < 1$, such that

$$S_j = \frac{N_j}{\ell_j} \approx (B^{2j})^{1-\eta} .$$

We finally define, for any integer $p \geq 1$,

$$A_{js;p} := \frac{1}{\ell_j} \sum_{k \in R_{j;s}} \beta_{jk}^p ,$$

and its corresponding estimator

$$\hat{A}_{js;p} = \frac{1}{\ell_j} \sum_{k \in R_{j;s}} \hat{\beta}_{jk}^p ,$$

similar to the ones suggested in [24], Remark 4.7.

Let us define the following weight function

$$w_{js;p} = I \left(\left| \hat{A}_{js;p} \right| > \kappa t_n^p \right) ;$$

we have:

$$f^* = \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \left(\sum_{k \in R_{j;s}} \hat{\beta}_{jk} \psi_{jk} \right) w_{js;p} , \quad (10)$$

where:

- J_n is the highest resolution level considered, taken such that

$$B^{J_n} = n^{\frac{1}{2}} ;$$

- κ is the threshold constant (for more discussions see for instance [4], [12]);
- the scaling factor t_n , depends on the size of the sample. We will fix

$$t_n = n^{-\frac{1}{2}} .$$

4 Minimax L^p -risk rates of convergence

Our main purpose is to describe the performance of the procedure and the optimality of its convergence rates with respect to general $L^p(S^2)$ -loss functions. This result is achieved in the next theorem.

Theorem 1 *Let $f \in \mathcal{B}_{\pi q}^r(G)$, the Besov ball so that $\|f\|_{\mathcal{B}_{\pi q}^r(G)} \leq M < +\infty$, $r - \frac{2}{\pi} > 0$. Consider f^* as defined by (10). For $p \in \mathbb{N}$, there exists a constant $c_p = c_p(p, r, q, M, B)$ such that*

$$\sup_{f \in \mathcal{B}_{\pi q}^r(G)} \mathbb{E} \|f^* - f\|_{L^p(S^2)}^p \leq c_p n^{-\alpha(r, \pi, p)} ,$$

where

$$\alpha(r, \pi, p) = \begin{cases} \frac{rp}{2(r+1)} & \text{for } \pi \geq \frac{2p}{2(r+2)} \\ \frac{p(r-2(\frac{1}{\pi}-\frac{1}{p}))}{2(r-2(\frac{1}{\pi}-\frac{1}{2}))} & \text{for } \pi < \frac{2p}{2(r+2)} \end{cases} .$$

If $p = +\infty$, there exists a constant $c_\infty = c_\infty(r, q, M, B)$

$$\sup_{f \in \mathcal{B}_{\pi q}^r(G)} \mathbb{E} \|f^* - f\|_\infty \leq c_\infty n^{-\alpha(r, \pi, p)} ,$$

where

$$\alpha(r, \pi, p) = \frac{(r - \frac{2}{\pi})}{2(r - 2(\frac{1}{\pi} - \frac{1}{2}))} .$$

Remark 2 This procedure achieves minimax the rates established in [4] and [12], see also [23].

5 Auxiliary Results

In order to prove Theorem 1, we will need the following

Lemma 3 Consider $\widehat{\beta}_{jk}$ as described in 8. There exist constants C_p, C_∞, C_A such that, for $B^j \leq n^{\frac{1}{2}}$, $j = 0, \dots, J_n$,

$$\mathbb{E} \left[\left| \widehat{\beta}_{jk} - \beta_{jk} \right|^p \right] \leq C_p n^{-p/2} , \quad p \geq 1 \quad (11)$$

$$\mathbb{E} \left[\sup_{k=1, \dots, N_j} \left| \widehat{\beta}_{jk} - \beta_{jk} \right|^p \right] \leq C_\infty (j+1)^p n^{-p/2} , \quad p \geq 1 , \quad (12)$$

and for all $\gamma > 0$ there exists $\kappa > 0$ such that

$$\mathbb{P} \left(\left| \widehat{A}_{js} - A_{js} \right| > \kappa t_n^p \right) \leq C_{p, \gamma} \frac{1}{n^\gamma} . \quad (13)$$

Proof. First of all, consider that, from Equations (8) and (9), we have

$$\begin{aligned} \mathbb{E} \left(\left| \widehat{\beta}_{jk} - \beta_{jk} \right|^p \right) &= \mathbb{E} (|\varepsilon_{jk; n}|^p) \\ &= (\text{Var}(\varepsilon_{jk; n}))^{\frac{p}{2}} \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \\ &= \frac{1}{n^{\frac{p}{2}}} \|\psi_{jk}\|_{L^2(S^2)}^p \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \\ &= O\left(n^{-\frac{p}{2}}\right) , \end{aligned}$$

to obtain (11). Now, for Mill's inequality, if $Z \sim N(0, 1)$, we have $\mathbb{P}(|Z| \geq x) \leq \sqrt{2/\pi} \exp(-x^2/2)/x$, which leads us, from (9), to

$$\begin{aligned} \mathbb{P}(|\varepsilon_{jk;n}| \geq x) &= 2\mathbb{P}(|Z| \geq \sqrt{nx}) \\ &\leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{nx}{2}}}{\sqrt{nx}} \\ &\leq C_\varepsilon e^{-\frac{nx}{2}}, \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{k=1, \dots, N_j} |\hat{\beta}_{jk} - \beta_{jk}|^p \right] &= \int_{\mathbb{R}^+} x^{p-1} \mathbb{P} \left(\sup_{k=1, \dots, N_j} |\hat{\beta}_{jk} - \beta_{jk}| \geq x \right) dx \\ &= \int_{\mathbb{R}^+} x^{p-1} \mathbb{P} \left(\sup_{k=1, \dots, N_j} |\varepsilon_{jk;n}| \geq x \right) dx \\ &= E_1 + E_2, \end{aligned}$$

where

$$\begin{aligned} E_1 &= \int_{0 \leq x \leq \frac{2\sqrt{2}}{\sqrt{N}}j} x^{p-1} dx, \\ E_2 &= C \int_{x > \frac{2\sqrt{2}}{\sqrt{N}}j} x^{p-1} B^{2j} \max_k \mathbb{P}(|\varepsilon_{jk;n}| \geq x) dx. \end{aligned}$$

We can easily see that

$$E_1 = C_1 j^p n^{-\frac{p}{2}},$$

while on the other hand, considering that for $x > 2\sqrt{2/n}j$

$$B^{2j} e^{-\frac{nx}{2}} \leq e^{-\frac{nx^2}{4} - \frac{nx^2}{4} + 2j} \leq e^{-\frac{nx^2}{4}},$$

we obtain

$$\begin{aligned} E_2 &\leq C \int_{x > \frac{2\sqrt{2}}{\sqrt{N}}j} x^{p-1} B^{2j} e^{-\frac{nx^2}{2}} dx \\ &\leq C_2 n^{-\frac{p}{2}}, \end{aligned}$$

so we achieve (12). It remains to prove (13), which corresponds to prove that for all $\gamma > 0$ there exists $\kappa > 0$ such that

$$\Pr \left\{ \left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \hat{\beta}_{jk}^p - E \hat{\beta}_{jk}^p \right)^{1/p} > \frac{\kappa}{\sqrt{n}} \right\} \leq C_{p,\gamma} \frac{1}{n^\gamma}.$$

We start from rewriting

$$\tilde{\beta}_{jk} = \sqrt{n} \beta_{jk} + \sqrt{n} \varepsilon_{jk;n} = \sqrt{n} \beta_{jk} + \varepsilon_{jk}$$

So we need to analyze terms of the form

$$\left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p + \frac{p\sqrt{n}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk} \varepsilon_{jk}^{p-1} + \dots + \frac{pn^{(p-1)/2}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} \right)^{1/p}. \quad (14)$$

We start by observing that:

$$\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} \leq \left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{2p-2} \right)^{1/2} \left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^2 \right)^{1/2}$$

Now we know that

$$\sum_{k=1}^{\ell_j} \beta_{jk}^{2p-2} \leq \sum_{k=1}^{N_j} \beta_{jk}^{2p-2} = O\left(B^{-js} B^{-j(1-\frac{1}{p-1})}\right) = O\left(B^{-js} B^{-j(\frac{p-2}{p-1})}\right).$$

On the other hand, by Lemma 4, it is easy to see that for all $p, \gamma > 0$, there exists $\kappa > 0$ such that

$$\Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} |\varepsilon_{jk}|^p > \kappa \right\} \leq \frac{C_{p,\gamma}}{\ell_j^{\gamma/2}}.$$

It is immediate to see that

$$\frac{pn^{(p-1)/2}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} \leq C \frac{n^{(p-1)/2}}{\ell_j^{\frac{\gamma+2}{2}}} B^{-j(\frac{p-2}{p-1}+s)};$$

by choosing suitable s and γ , we have

$$\frac{pn^{(p-1)/2}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} = o\left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p\right).$$

The same holds for all the other mixed terms in Equation (14). ■

Lemma 4 Assume that $E\varepsilon_{jk} = 0$, $E\varepsilon_{jk}^2 = 1$, and

$$E\varepsilon_{jk_1} \varepsilon_{jk_2} \leq \frac{C_M}{\{1 + B^j d(\xi_{jk_1}, \xi_{jk_2})\}^M}, \text{ for all } M > 0.$$

Then for all $p \in \mathbb{N}$, $\gamma > 0$ there exists $\kappa > 0$ such that

$$\Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} |\varepsilon_{jk}|^p > \kappa \right\} \leq \frac{C_{p,\gamma}}{\ell_j^{\gamma/2}}.$$

Proof. Without loss of generality we can take p to be even; note indeed that

$$\Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} |\varepsilon_{jk}|^p > \kappa \right\} \leq \Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^{2p} > \kappa^2 \right\}.$$

Now we can write

$$\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p = E\varepsilon_{jk}^p + \sum_{\tau=1}^p c_\tau H_\tau(\varepsilon_{jk}),$$

whence

$$\Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p > p(\kappa + E\varepsilon_{jk}^p) \right\} \leq \sum_{\tau=1}^p \Pr \left\{ \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} H_\tau(\varepsilon_{jk}) > \frac{\kappa}{c_\tau} \right\}.$$

By a simple application of the Markov's inequality, the result will hence follow if we simply show that

$$E \left[\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} H_\tau(\varepsilon_{jk}) \right]^\gamma \leq \frac{C}{\ell_j^{\gamma/2}}.$$

Now let us take for notational simplicity $\tau = 2$; the argument for the other terms is identical. We have

$$\begin{aligned} E \left[\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} H_\tau(\varepsilon_{jk}) \right]^\gamma &= \frac{1}{\ell_j^\gamma} \sum_{k_1, \dots, k_\gamma=1}^{\ell_j} E \{ H_\tau(\varepsilon_{jk_1}) \dots H_\tau(\varepsilon_{jk_\gamma}) \} \\ &= \frac{1}{\ell_j^\gamma} \left\{ \sum_{k_1 k_2}^{\ell_j} [E(\varepsilon_{jk_1} \varepsilon_{jk_2})]^2 \right\}^{\gamma/2} \\ &+ \frac{1}{\ell_j^\gamma} \left\{ \sum_{k_1 k_2}^{\ell_j} [E(\varepsilon_{jk_1} \varepsilon_{jk_2})]^2 \right\}^{\frac{\gamma}{2}-2} \left\{ \sum_{k_1 \dots k_4}^{\ell_j} E(\varepsilon_{jk_1} \varepsilon_{jk_2}) E(\varepsilon_{jk_2} \varepsilon_{jk_3}) E(\varepsilon_{jk_3} \varepsilon_{jk_4}) E(\varepsilon_{jk_4} \varepsilon_{jk_1}) \right\} \\ &+ \frac{1}{\ell_j^\gamma} \left\{ \sum_{k_1 k_2}^{\ell_j} [E(\varepsilon_{jk_1} \varepsilon_{jk_2})]^2 \right\}^{\frac{\gamma}{2}-4} \left\{ \sum_{k_1 \dots k_6}^{\ell_j} E(\varepsilon_{jk_1} \varepsilon_{jk_2}) \dots E(\varepsilon_{jk_6} \varepsilon_{jk_1}) \right\} \\ &+ \frac{1}{\ell_j^\gamma} \left\{ \sum_{k_1 \dots k_\gamma}^{\ell_j} E(\varepsilon_{jk_1} \varepsilon_{jk_2}) \dots E(\varepsilon_{jk_\gamma} \varepsilon_{jk_1}) \right\} \\ &= O(\ell_j^{-\gamma/2}) + O(\ell_j^{-\frac{\gamma}{2}-1}) + \dots + O(\ell_j^{-\gamma+1}), \end{aligned}$$

because

$$\begin{aligned} \sum_{k_1 \dots k_\gamma} E(\varepsilon_{jk_1} \varepsilon_{jk_2}) \dots E(\varepsilon_{jk_\gamma} \varepsilon_{jk_1}) &\leq \sum_{k_1 \dots k_\gamma} |E(\varepsilon_{jk_1} \varepsilon_{jk_2})| \dots |E(\varepsilon_{jk_{\gamma-1}} \varepsilon_{jk_\gamma})| \\ &\leq \ell_j \left\{ \sum_{k_2}^{\ell_j} |E(\varepsilon_{jk_1} \varepsilon_{jk_2})| \right\}^{\gamma-1} = O(\ell_j) . \end{aligned}$$

■

6 Proof of Theorem 1 (upper bound)

First of all, we stress that where explicitly mentioned, proofs will be totally similar to the standard thresholding case described in [4], hence we will omit them. Considering that $\sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} = \sum_{k=1}^{N_j}$, it is easy to see that

$$\begin{aligned} \mathbb{E} \|f^* - f\|_{L^p(\mathbb{S}^2)}^p &= \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \left(\sum_{k \in R_{j;s}} \widehat{\beta}_{jk} \psi_{jk} \right) w_{js} - \sum_{j \geq 0} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \\ &= \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} (w_{js} \widehat{\beta}_{jk} - \beta_{jk}) \psi_{jk} - \sum_{j > J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \\ &\leq 2^{p-1} \left(\mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} (w_{js} \widehat{\beta}_{jk} - \beta_{jk}) \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p + \left\| \sum_{j > J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \right) \\ &=: I + II . \end{aligned}$$

CASE I: Regular Zone

Consider $p < +\infty$. For $p \leq \pi$, we have $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^r$: we can therefore take $\pi = p$. Consider instead the case $p > \pi$: we use the embedding $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^{r-\frac{2}{p}+\frac{2}{\pi}}$, and moreover we assume

$$r \geq \frac{2}{p}, \frac{r}{2r+2} = \frac{rp}{2(r+1)p} \leq \frac{r\pi}{2p} ,$$

we have as in [4] that

$$II \leq O \left(n^{-\frac{pr}{2r+2}} \right) ,$$

as claimed.

Consider now the variance term. First of all, we easily obtain from the Loeve

inequality

$$\begin{aligned}
I &\leq C \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \\
&\leq C J_n^{p-1} \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p.
\end{aligned}$$

As described in [4], see also [34], we have the following needlelet property:

$$\mathbb{E} \left\| \sum_k \alpha_k \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p = \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \sum_k \mathbb{E} \|\alpha_k\|_{L^p(\mathbb{S}^2)}^p.$$

Hence we have

$$\begin{aligned}
&\sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \\
&= \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left(|\hat{A}_{js}| \geq t_n^p \right) I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right\|_{L^p(\mathbb{S}^2)}^p \\
&\quad + \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left(|\hat{A}_{js}| \geq t_n^p \right) I \left(|A_{js}| < \frac{t_n^p}{2} \right) \right\|_{L^p(\mathbb{S}^2)}^p \\
&\quad + \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left(|\hat{A}_{js}| < t_n^p \right) I \left(|A_{js}| \geq 2t_n^p \right) \right\|_{L^p(\mathbb{S}^2)}^p \\
&\quad + \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \hat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left(|\hat{A}_{js}| < t_n^p \right) I \left(|A_{js}| < 2t_n^p \right) \right\|_{L^p(\mathbb{S}^2)}^p \\
&\leq C \left\{ \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \mathbb{E} \left[\left(\hat{\beta}_{jk} - \beta_{jk} \right)^p I \left(|\hat{A}_{js}| \geq t_n^p \right) I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right] \right. \\
&\quad + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \mathbb{E} \left[\left(\hat{\beta}_{jk} - \beta_{jk} \right)^p I \left(|\hat{A}_{js}| \geq t_n^p \right) I \left(|A_{js}| < \frac{t_n^p}{2} \right) \right] \\
&\quad + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p |\beta_{jk}|^p \mathbb{E} \left[I \left(|\hat{A}_{js}| < t_n^p \right) I \left(|A_{js}| \geq 2t_n^p \right) \right] \\
&\quad \left. + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p |\beta_{jk}|^p \mathbb{E} \left[I \left(|\hat{A}_{js}| < t_n^p \right) I \left(|A_{js}| < 2t_n^p \right) \right] \right\}
\end{aligned}$$

$$= Aa + Au + Ua + Uu .$$

The idea now is to split this sum into four terms: in one of them, Aa , both the \hat{A}_{js} and A_{js} are large; in another one, Uu , they are both small and in the last two of them, Au and Ua , the distance between \hat{A}_{js} and A_{js} is large. In the first two cases, in order to achieve the minimax rate of convergence, we will split the sum into two parts and we will show the convergence of each term by using mainly (4), (11) and (12). The convergence of the last two terms will be proved by using (13). We start by studying

$$\begin{aligned} Aa &\leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \mathbb{E} \left[\left| \hat{\beta}_{jk} - \beta_{jk} \right|^p \right] I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \\ &\leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \mathbb{E} \left[\left| \hat{\beta}_{jk} - \beta_{jk} \right|^p \right] . \end{aligned}$$

As in [4], we fix J_{1n} such that

$$B^{J_{1n}} = O \left(n^{\frac{1}{2(r+1)}} \right) ,$$

hence we have

$$\begin{aligned} Aa &\leq Cn^{-p/2} \left(\sum_{j \leq J_{1n}} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) + \sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right) \\ &\leq Cn^{-p/2} \left(\sum_{j \leq J_{1n}} (S_j \cdot \ell_j) B^{j(p-2)} + \sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right) \\ &\leq Cn^{-p/2} \left(\sum_{j \leq J_{1n}} B^{jp} + \sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right) \\ &\leq Cn^{-p/2} \left(B^{pJ_{1n}} + \sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \right) . \end{aligned}$$

Observe now that

$$\begin{aligned}
\sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{j(p-2)} I\left(|A_{js}| \geq \frac{t_n^p}{2}\right) &\leq \sum_{j=J_{1n}}^{J_n} \ell_j B^{j(p-2)} \sum_{s=1}^{S_j} |A_{js}| \left(\frac{t_n^p}{2}\right)^{-1} \\
&\leq C t_n^{-p} \sum_{j=J_{1n}}^{J_n} \ell_j B^{j(p-2)} \sum_{s=1}^{S_j} \frac{1}{\ell_j} \sum_{k \in R_{j;s}} |\beta_{jk}|^p \\
&\leq C t_n^{-p} \sum_{j=J_{1n}}^{J_n} B^{j(p-2)} \sum_{k=1}^{N_j} |\beta_{jk}|^p \\
&\leq C n^{\frac{p}{2}} \sum_{j=J_{1n}}^{J_n} \sum_{k=1}^{N_j} |\beta_{jk}|^p B^{j(p-2)} .
\end{aligned}$$

Now, because $f \in B_{pq}^r$, we have

$$\sum_{k=1}^{N_j} |\beta_{jk}|^p B^{j(p-2)} = C \sum_{k=1}^{N_j} |\beta_{jk}|^p \|\psi_{jk}\|_p^p \leq C B^{-prj} .$$

Hence, as in [4]

$$n^{\frac{p}{2}} \sum_{j=J_{1n}}^{J_n} \sum_{k=1}^{N_j} |\beta_{jk}|^p B^{j(p-2)} \leq C n^{\frac{p}{2(r+1)}} \leq B^{pJ_{1n}} .$$

Finally

$$Aa \leq C n^{-p/2} B^{pJ_{1n}} = C n^{\frac{-pr}{2(r+1)}} .$$

Consider now the term Uu . We have

$$\begin{aligned}
Uu &\leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p |\beta_{jk}|^p I(|A_{js}| < 2t_n^p) \\
&\leq C \sum_{j \leq J_n} B^{j(p-2)} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} |\beta_{jk}|^p I(|A_{js}| < 2t_n^p) \\
&\leq C \sum_{j \leq J_n} \ell_j B^{j(p-2)} \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) \\
&\leq C \left[\sum_{j \leq J_{1n}} N_j B^{j(p-2)} 2t_n^p + \sum_{j=J_{1n}}^{J_n} \ell_j B^{j(p-2)} \sum_{s=1}^{S_j} A_{js} \right] \\
&\leq C \left[\sum_{j \leq J_{1n}} N_j B^{j(p-2)} n^{-\frac{p}{2}} + \sum_{j=J_{1n}}^{J_n} \sum_{k=1}^{N_j} |\beta_{jk}|^p \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \right] \\
&\leq C \left[n^{-\frac{p}{2}} B^{pJ_{1n}} + B^{-prJ_{1n}} \right] = O\left(n^{-\frac{pr}{2(r+1)}}\right) .
\end{aligned}$$

Let us study now Au and Ua . As in [4], we have

$$\begin{aligned}
Au &\leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \left(\mathbb{E} \left[|\hat{\beta}_{jk} - \beta_{jk}|^{2p} \right] \right)^{\frac{1}{2}} \left(\mathbb{P} \left(|\hat{A}_{js} - A_{js}| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) \right)^{\frac{1}{2}} \\
&\leq CB^{pJ_n} n^{-\frac{p}{2}} n^{-\gamma} \leq Cn^{-\gamma} ; \\
Ua &\leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p |\beta_{jk}|^p \left(\mathbb{P} \left(|\hat{A}_{js} - A_{js}| \geq \kappa n^{-\frac{p}{2}} \right) \right) \leq Cn^{-\gamma} \|F\|_p^p .
\end{aligned}$$

Because for $r \geq 1$, we have

$$n^{-\gamma} \leq n^{-\frac{1}{2}} \leq n^{\frac{-r}{2(r+1)}} ,$$

the result is proved.

Consider now $p = +\infty$: we assume now $f \in \mathcal{B}_{\infty, \infty}^r$, to obtain

$$\mathbb{E} \|f^* - f\|_{\infty} \leq \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{k=1}^{N_j} (w_j \hat{\beta}_{jk} - \beta_{jk}) \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^2)} + \left\| \sum_{j > J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^2)} = I + II .$$

As in [4], we have:

$$II = O \left(n^{-\frac{r}{2(r+1)}} \right) .$$

For what concerns I , we instead have

$$\begin{aligned}
I &\leq \sum_{j=0}^{J_n} \mathbb{E} \left\| \sum_{k=1}^{N_j} (w_j \hat{\beta}_{jk} - \beta_{jk}) \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^2)} \leq C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[\sup_k (w_j \hat{\beta}_{jk} - \beta_{jk}) \right] \\
&\leq C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[\sup_k (\hat{\beta}_{jk} - \beta_{jk}) \right] I \left(|A_j| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) \\
&\quad + C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[\sup_k (\hat{\beta}_{jk} - \beta_{jk}) \right] I \left(|\hat{A}_j - A_j| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) \\
&\quad + C \sum_{j=0}^{J_n} B^j \sup_k |\beta_{jk}| \mathbb{E} \left[I \left(|\hat{A}_j - A_j| \geq \kappa n^{-\frac{p}{2}} \right) \right] \\
&\quad + C \sum_{j=0}^{J_n} B^j \sup_k |\beta_{jk}| I \left(|A_j| < 2\kappa n^{-\frac{p}{2}} \right) \\
&= Aa + Au + Ua + Uu .
\end{aligned}$$

Again, we choose $J_{1,n}$ such that

$$B^{J_{1,n}} = \kappa' n^{\frac{1}{2(r+1)}} ; \quad I \left(|A_j| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) = 0 \text{ for } j > J_{1,n} ,$$

and similarly to [4], we obtain

$$\begin{aligned} Aa &\leq C J_{1,n} n^{-\frac{1}{2}} B^{J_{1,n}} \leq C n^{-\frac{r}{2(r+1)}} ; \\ Uu &\leq C \left\{ B^{-J_{1,n}(r+1)} + B^{-J_{1,n}} \right\} \leq C n^{-\frac{r}{2(r+1)}} . \end{aligned}$$

The other two terms Au and Ua are similar to the case previously described. For general π and q , we observe that $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{\infty}^{r'}$, $r' = r - 2/\pi$. Hence we obtain

$$\mathbb{E} \|f^* - f\|_{L^\infty(\mathbb{S}^2)} \leq C J_n n^{-\frac{r'}{2(r'+1)}} = C J_n n^{-\frac{r-2/\pi}{2(r-2(1/\pi-1/2))}} ,$$

as claimed.

CASE II: Sparse Zone

The proof follows strictly the same steps of the regular case. Indeed we have $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^{r-2(\frac{1}{\pi}-\frac{1}{p})}$. Hence we have

$$\begin{aligned} \mathbb{E} \|f^* - f\|_{L^p(\mathbb{S}^2)}^p &\leq 2^{p-1} \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left(w_{js} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p + \left\| \sum_{j > J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^p(\mathbb{S}^2)}^p \\ &= : I + II . \end{aligned}$$

Also in this case, as in [4], because $r - 2/\pi + 1 \geq 1$, we have for the bias term:

$$II = O \left(n^{-(r-2(\frac{1}{\pi}-\frac{1}{p}))/2(r-2(\frac{1}{\pi}-\frac{1}{p}))} \right) .$$

On the other hand, we split I again into four terms as above. On one hand, we obtain

$$\begin{aligned} Au &\leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p \left(\mathbb{E} \left[\left| \widehat{\beta}_{jk} - \beta_{jk} \right|^{2p} \right] \right)^{\frac{1}{2}} \left(\mathbb{P} \left(\left| \widehat{A}_j - A_j \right| \geq \frac{\kappa t_n^p}{2} \right) \right)^{\frac{1}{2}} \\ Ua &\leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \|\psi_{jk}\|_{L^p(\mathbb{S}^2)}^p |\beta_{jk}|^p \left(\mathbb{P} \left(\left| \widehat{A}_j - A_j \right| \geq \kappa t_n^p \right) \right) , \end{aligned}$$

whose upper bounds recall exactly the same procedure developed in regular zone. On the other hand, consider initially:

$$Aa \leq C n^{-\frac{p}{2}} \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{j(p-2)} I \left(|A_{js}| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) .$$

In this case, we fix J_{2n} so that

$$B^{J_{2n}} = O \left(n^{\frac{1}{2(r-\frac{2}{\pi}+1)}} \right) , I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \equiv 0 \text{ for } j \geq J_{2n} .$$

to obtain

$$\begin{aligned}
Aa &\leq Cn^{-\frac{p}{2}} \sum_{j \leq J_{2n}} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{j(p-2)} I \left(|A_{js}| \geq \frac{t_n^p}{2} \right) \\
&\leq Cn^{-\frac{p}{2}} \sum_{j \leq J_{2n}} B^{j(p-2)} N_j \frac{|A_j|^{\frac{1}{p}}}{t_n} \\
&\leq Cn^{-\frac{p}{2}} t_n^{-1} \sum_{j \leq J_{2n}} B^{j(p-2)} N_j^{1-\frac{1}{p}} \left(\sum_{k=1}^{N_j} |\beta_{jk}|^p \right)^{\frac{1}{p}} \\
&\leq Cn^{-\frac{p}{2}} t_n^{-1} \sum_{j \leq J_{2n}} B^{j(p-2)} N_j^{1-\frac{1}{p}} \left(\sum_{k=1}^{N_j} |\beta_{jk}|^\pi \right)^{\frac{1}{\pi}} \\
&\leq Cn^{-\frac{p}{2}} t_n^{-1} \sum_{j \leq J_{2n}} B^{j(p-2)} N_j^{1-\frac{1}{p}} \left(\sum_{k=1}^{N_j} |\beta_{jk}|^\pi \|\psi_{jk}\|_{L^\pi(\mathbb{S}^2)}^\pi \right)^{\frac{1}{\pi}} B^{-j(1-\frac{2}{\pi})} \\
&\leq Cn^{-\frac{p}{2}} t_n^{-1} \sum_{j \leq J_{2n}} B^{j(p-2)} B^{2j(1-\frac{1}{p})} B^{-rj} B^{-j(1-\frac{2}{\pi})} \\
&\leq Cn^{\frac{1-p}{2}} B^{J_{2n}(p-2-(r+1-\frac{2}{\pi}))} \\
&\leq Cn^{-\frac{p(r+2(\frac{1}{p}-\frac{1}{\pi}))}{2(r-\frac{2}{\pi}+1)}}
\end{aligned} \tag{15}$$

because in the last inequality we used

$$\frac{1-p}{2} + \frac{(p-2-(r+1-\frac{2}{\pi}))}{2(r-\frac{2}{\pi}+1)} = -\frac{p(r+2(\frac{1}{p}-\frac{1}{\pi}))}{2(r-\frac{2}{\pi}+1)}.$$

Consider now

$$\begin{aligned}
Uu &\leq C \sum_{j \leq J_n} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) = C \sum_{j \leq J_{2n}} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) \\
&+ \sum_{j=J_{2n}}^{J_n} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) = Uu_1 + Uu_2.
\end{aligned}$$

Now fix

$$m = \frac{p-2}{r-\frac{2}{\pi}+1},$$

so that

$$\begin{aligned}
p-m &= p \frac{r-2(\frac{1}{\pi}+\frac{1}{p})}{r-\frac{2}{\pi}+1} > 0; \\
m-\pi &= \frac{p-\pi(r+1)}{r-\frac{2}{\pi}+1} > 0.
\end{aligned}$$

Simple calculations lead to

$$\begin{aligned}
Uu_1 &= C \sum_{j \leq J_{2n}} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) \\
&\leq C \sum_{j \leq J_{2n}} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} \left(\frac{A_{js}}{t_n^p} \right)^{\frac{1}{p}-1} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js}^{\frac{1}{p}} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-2)} \ell_j^{1-\frac{1}{p}} \sum_{s=1}^{S_j} \left(\sum_{k \in R_{js}} \beta_{jk}^p \right)^{\frac{1}{p}} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-2)} (\ell_j \cdot S_j)^{1-\frac{1}{p}} S_j^{\frac{1}{p}} \left(\sum_{k=1}^{N_j} \beta_{jk}^m \right)^{\frac{1}{m}} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-2)} N_j \left(\sum_{k=1}^{N_j} |\beta_{jk}|^m \|\psi_{jk}\|_{L^m(\mathbb{S}^2)}^m \right)^{\frac{1}{m}} B^{-j(1-\frac{2}{m})} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-1+\frac{2}{m})} B^{-jr} \\
&\leq Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-1+\frac{2}{m})} B^{-j(r-\frac{2}{\pi}+\frac{2}{m})} \\
&= Ct_n^{p-1} \sum_{j \leq J_{2n}} B^{j(p-2-(r-1-\frac{2}{\pi}))} \\
&= Ct_n^{p-1} B^{J_{2n}(p-2-(r-1-\frac{2}{\pi}))} \\
&\leq Cn^{\frac{1-p}{2}} B^{J_{2n}(p-2-(r-1-\frac{2}{\pi}))} ,
\end{aligned}$$

exactly as above in Equation (15). We have to study just the last term

$$Uu_2 = C \sum_{j=J_{2n}}^{J_n} N_j B^{j(p-2)} A_j I(|A_j| < 2t_n^p) .$$

We have

$$\begin{aligned}
Uu_2 &= C \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js} I(|A_{js}| < 2t_n^p) \\
&\leq C \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js}^{\frac{m}{p}} A_{js}^{1-\frac{m}{p}} I(|A_j| < 2t_n^p) \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} \ell_j \sum_{s=1}^{S_j} A_{js}^{\frac{m}{p}} \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} \ell_j^{1-\frac{m}{p}} \sum_{s=1}^{S_j} \left(\sum_{k \in R_{js}} |\beta_{jk}|^p \right)^{\frac{1}{p} \cdot m} \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} \ell_j S_j \left(\sum_{k=1}^{N_j} |\beta_{jk}|^p \right)^{\frac{1}{p} \cdot m} \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} (\ell_j \cdot S_j) \left(\sum_k |\beta_{jk}|^m \right) \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-2)} N_j \left(\sum_k |\beta_{jk}|^m \right) \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-m)} \left(\sum_k |\beta_{jk}|^m \|\psi_{jk}\|_{L^m(\mathbb{S}^2)}^m \right) \\
&\leq Ct_n^{p-m} \sum_{j=J_{2n}}^{J_n} B_j^{j(p-m)} B^{-jm(r-\frac{2}{\pi}+\frac{2}{m})} .
\end{aligned}$$

We can easily see that

$$(p-m) - m \left(r - \frac{2}{\pi} + \frac{2}{m} \right) = p - 2 - \frac{p-2}{r-\frac{2}{\pi}+1} \left(1 + r - \frac{2}{\pi} \right) = 0 .$$

Hence

$$Uu_2 \leq Ct_n^{p-n} = O \left(n^{-\frac{p(r+2(\frac{1}{p}-\frac{1}{\pi}))}{2(r-\frac{2}{\pi}+1)}} \right) ,$$

as claimed.

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